Mathematics Bootcamp
Master in Economics

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Contents

1 Introduction

This short guide is an introduction to the mathematics that you will study in the course "Advanced Mathematics for Economic Analysis". Its purpose is to make you familiar with some concepts that will be used in that course. However, a secondary purpose is to offer a way how to think deeper about mathematics. In many Bachelor's programs, you learn how to apply certain tools, but the more you study, the more you have to prove statements and understand how the concepts work.

Since this guide is about mathematics, there will be lots of symbols, however, I try to focus on intuition rather than the formal definitions (we will work more on the latter in the course). Personally, I need to understand both the mathematical details and the idea behind them to fully grasp any concept.

Before starting the exposition, I would like to share a metaphor that has helped me understand mathematics better. When I started graduate studies, I thought of mathematics as concepts you learn and then use to solve problems. Nowadays, I tend to think of it as a game you play, where you have to learn about the pieces and the rules they obey. Just like chess, mathematics is all about learning about the different pieces and how they are allowed to move – and what happens when they interact.

These notes are developed as a revision of Emil Bustos's notes from the summer of 2020, which draw heavily from other material, mostly Mark Voorneveld's lecture notes. They are excellent at combining clarity with intuition. If you read them instead, there is not much I can add. With that said, I still hope this guide can be useful in understanding these topics at least somewhat better.

Johan Orrenius, Summer of 2024

2 Sets and Numbers

In this section, we learn about sets and numbers. These are the bits and pieces of mathematics and the rules for combining them are the foundations for the other areas we will discuss. This section draws heavily from *Appendix A* in Mark's lecture notes.

2.1 What Are Sets

Sets are collections of elements, for example, $\{1, 2, 3\}$ or $\{a, b\}$. The elements can be all sorts of things, such as numbers, functions, names, or other sets. When we describe sets, we either list the elements directly, such as in the examples above. Another way is to define the condition for an element to be part of a set.

As an example: the set of all capital cities starting with the letter O can either be written as any of the two:

$$\{Oslo, Ottawa, Ougadougou\}$$

$$\{Capital\ city : Capital\ city\ starts\ with\ O\}$$

2.2 Set Operations

Now that we have defined sets, we can start discussing relations between sets. Conceptually, we can look at parts of sets, join sets together and look at the parts sets have in common.

First, we need some more notation. To say that element x belongs to the set X, we write $x \in X$. If an element x is not in X we write $x \notin X$. Further, if a statement applies to all elements in X we say $\forall x, x \in X$, and if it applies to at least one element we say $\exists x, x \in X$. All statements can be negated by $\neg$. Now, consider two arbitrary sets A and B.

- If every element in A also belongs to B, then A is a subset of B, written as $A \subseteq B$
- If every element in A belongs to B and conversely, then the sets A and B are identical, written as $A = B$
- If A is a subset of B but B is not a subset of A, then we say that A is a proper subset of B, written as $A \subset B$

Next, we discuss elements that are in either A or B, or both.

- The set of elements that are either in A or in B (or both) is called the union of A and B, written as $A \cup B$
- Likewise, if we have n sets $A_1, \ldots, A_n$, their union is denoted $\cup_{i=1}^{n} A_i$ or $A_1 \cup \cdots \cup A_n$ and consists of all elements that belong to one or more of these sets.

- The set of elements that are in A and in B is called the intersection of A and B, written as $A \cap B$
- The set of elements in A that are not in B is written $A \setminus B$
- Assume $A \subset X$, then we say that the complement of A is $X \setminus A$, and it is written as A^C

2.3 What Are Numbers

We are used to working with numbers to count and label things. The ordinary numbers $1, 2, 3, ...$ are called the natural numbers. We denote the sets of all natural numbers as $\mathbb{N}$.

Next, we extend the natural numbers to include the negatives and zero, that is the set $\cup_i^\infty \{-i, i\} \cup \{0\}$.
These are called the integers and are denoted as $\mathbb{Z}$.

We then extend the integers and allow fractions, such as $\frac{1}{2}$. Formally, we take all numbers that can be written on the form $\frac{p}{q}$, where p, q are integers and $q \neq 0$. We call this set the rationals and denote it $\mathbb{Q}$.

Now we get to the workhorse numbers of this course, the real numbers. Numbers such as π and $\sqrt{2}$ are not included in the rationals but are real numbers. Think of them as a way to "fill out" or "complete" the holes between rational numbers. Take a rational number and start adding more decimals: 3, 3.1, 3.14, 3.141, 3.1415, and so on. For any such sequence, there will be convergence to a unique value, a real number! The real numbers are denoted by $\mathbb{R}$.

2.4 Properties of the Real Numbers

The real numbers have many interesting properties. One of the key properties is the "least upper bound property". To discuss this, we need to informally introduce the idea of boundedness. A set is bounded from above if we can find a number that is larger than all elements of the set.

As an example, consider the unit interval $[0, 1] = \{x \in \mathbb{R} : 0 \leq x \leq 1\}$. All elements in this set are smaller than or equal to 1. The same is true for the open interval $(0, 1) = \{x \in \mathbb{R} : 0 < x < 1\}$. In the second case, 1 is not included in the interval, but we can still bind the set by that number. We note that the same is true for any real number larger than 1: any number in the two intervals is smaller than 2 or 6 as well. Thus, these numbers are all upper bounds to the intervals.

More generally, for any non-empty set of real numbers that is bounded, there exists one real number which is the smallest real number that bounds the set. This number is called the least upper bound, or the supremum. If the supremum is included in the set, such as the case with $[0, 1]$, it is called the maximum value.

3 Linear Algebra

3.1 What are Vectors

As we recently learned, sets are collections of elements. Notably, they are unordered collections: the set $\{a, b\}$ is equal to the set $\{b, a\}$. In contrast, we can define ordered pairs when we want the order of elements to matter, such as coordinates. For example, consider the ordered pair (a, b).

You've probably seen vectors previously, maybe as a force that has both a magnitude and a direction. To go from ordered pairs to vectors, we need to impose some rules on how to combine ordered pairs. We can imagine all kinds of rules on how to combine ordered pairs, however, we will focus on a very specific kind. Take a set (real numbers in this case) and give the ordered pairs two operations. This means that we can add ordered pairs pairwise, e.g. $(1, 3) + (2, 5) = (3, 8)$ and that we can scale them, e.g. $2 * (1, 4) = (2, 8)$. We then say we have a vector space. The ordered pairs, in our case objects of the form, (a, b) with $a, b \in \mathbb{R}$, are called vectors.

We can also visualize vectors. Consider the vector $\mathbf{v}_1 = (1, 1, 1)$: it can be drawn as a line in 3-dimensional space that goes one step in each direction. Let's return to our two operations: addition and multiplication. If we take the other vector $\mathbf{v}_2 = (1, 2, 3)$ and add it to $\mathbf{v}_1$, then we get the new vector $\mathbf{v}_3 = \mathbf{v}_1 + \mathbf{v}_2 = (1, 1, 1) + (1, 2, 3) = (2, 3, 4)$, which we draw as well. Next, we do multiplication: if we multiply $\mathbf{v}_1$ by 2, we get the vector $2\mathbf{v}_1 = 2 * (1, 1, 1) = (2, 2, 2)$, which is the original vector, but making it go twice as far!

In the same way that we can measure the absolute value of a one-dimensional number. We can extend the same concept to arbitrary dimensions and define it as $||v|| = \sqrt{\sum_i^n v_i^2}$. This is the Euclidian norm. For example $||\mathbf{v}_3|| = \sqrt{2^2 + 3^2 + 4^2} = \sqrt{4 + 9 + 16} = \sqrt{29}$. The vectors are illustrated in Figure 1

3.2 Matrices

You've likely seen matrices before, which are collections of rows and columns organized in a rectangular fashion. We often denote matrices by bold capital letters, $\mathbf{A}$, and its element in row i and column j as a_{ij}.

If you have two matrices of the same dimension you can add them element-by-element. For example:

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix}$$

Moreover, we can multiply the elements of a matrix with a scalar:

$$5 * \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 5 \\ 5 & 5 \end{bmatrix}$$

These are similar to the rules that we used to combine vectors. This is no coincidence since matrices are vector spaces.

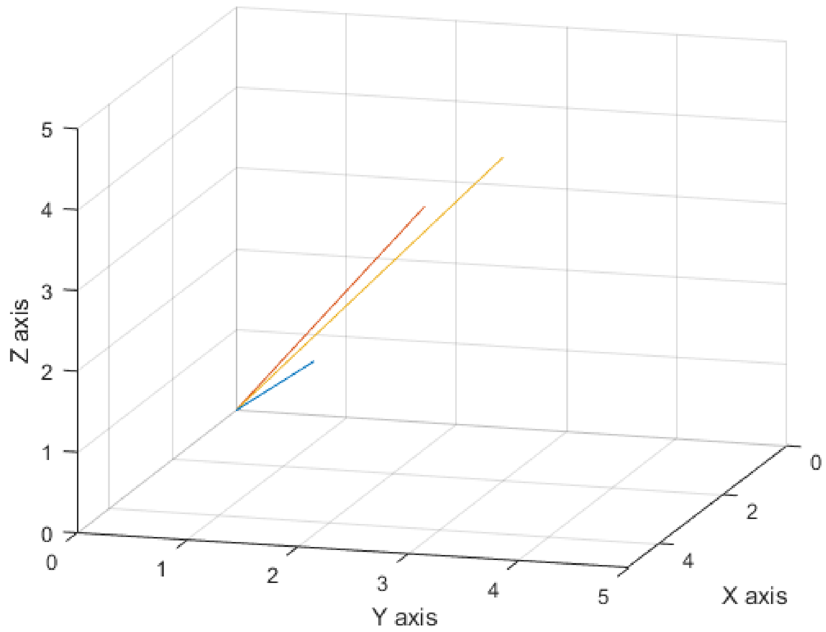

Figure 1: Here we draw the vectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

3.3 Systems of Linear Equations

A related topic is the study of systems of linear equations. These notes draw heavily from Strang's book Linear Algebra and its Applications". A system of linear equations can have one, no, or an infinite number of solutions.

3.3.1 One solution

The following system has one (unique) solution $(2, 3)$:

$$\begin{cases} 2x - y = 1 \\ x + y = 5 \end{cases}$$

If we think of this as two lines in the plane, they are given by $y = 2x - 1$ and $y = 5 - x$. They have one intersection (draw!).

3.3.2 No solution

The following system has no solution:

$$\begin{cases} x = 0.5y \\ y - 2x = 1 \end{cases}$$

If you draw the lines, you see that they are parallel.

3.3.3 Infinity of Solutions

Finally, we can have an infinite amount of solutions

$$\begin{cases} x = 0.5y \\ y - 2x = 0 \end{cases}$$

If you draw the lines, you see that they are on top of each other.

Another way to think of these systems is to think of them as vectors in $\mathbb{R}^2$. The system:

$$\begin{cases} 2x - y = 1 \\ x + y = 5 \end{cases}$$

can be re-written as:

$$x \begin{pmatrix} 2 \\ 1 \end{pmatrix} + y \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$$

Now, the question is: which combinations of (x, y) give us the right-hand side? If the answer is unique, we're in case 1 from before. If there is no solution, we're in case 2. If there is an infinite amount, we're in case 3.

By the way, this is asking for linear combinations of vectors. Moreover, we can add scalars to vectors and add them together. Notably, we're working in a real vector space!

3.4 Gaussian Elimination

A neat way of solving systems of linear equations is to use Gaussian elimination. We illustrate the method with an example:

$$\begin{cases} 2u + v + w = 5 \\ 4u - 6v = -2 \\ -2u + 7v + 2w = 9 \end{cases}$$

We put this in matrix form:

$$\begin{pmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \\ 9 \end{pmatrix}$$

As of now, all lines (except the second one) have all variables (u, v, w). Our idea is to see if we can transform the system to have w in the third line, v and w in the second line, and all three in the first line. Then our system would be much easier to solve.

When we do Gaussian elimination, we are allowed to perform three operations. These operations preserve the solution to the system of linear equations.

- Add a multiple of one row to another row
- Multiply all entries in a row with a non-zero number
- Exchanging two rows

3.4.1 Unique Solution

First, we continue with our example from before and see that we have a unique solution.

$$\begin{pmatrix} 2 & 1 & 1 & 5 \\ 4 & -6 & 0 & -2 \\ -2 & 7 & 2 & 9 \end{pmatrix} \sim \begin{pmatrix} 2 & 1 & 1 & 5 \\ 0 & -8 & -2 & -12 \\ 0 & 8 & 3 & 14 \end{pmatrix} \sim \begin{pmatrix} 2 & 0 & 3/4 & 7/2 \\ 0 & 1 & 1/4 & 3/2 \\ 0 & 0 & 1 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

This gives us the solution $(u, v, w) = (1, 1, 2)$.

3.4.2 No solution

Next, we go on to an example with no solution. If you do these operations and you end up with a row with only zeroes on the left and a non-zero number to the right, then you have an inconsistent system.

$$\begin{pmatrix} 1 & 0 & 5 \\ 0 & 0 & 1 \end{pmatrix}$$

Then $x = 5$ and we have no solution for y. Thus, there are no solutions to the system.

3.4.3 Infinity of Solutions

Finally, we look at an example with an infinity of solutions. If you do these operations and you end up with a row of all zeroes, then you have a system with an infinite amount of solutions. This implies that the original vectors are linearly independent.

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 1 & 1 & 8 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & -1 & -1 & -4 \\ 0 & 0 & 1 & 1 & 1 & 8 \end{pmatrix}$$

Then we have columns with pivot variables, which are the columns with leading one and all other elements equal to zero. In our case, they are columns 1 and 3. Thus, we set $(x_2, x_4, x_5) = (t, \theta, \varphi) \in \mathbb{R}^3$. Accordingly, we back out (x_1, x_3) as:

$$x_1 = -4 + \theta + \varphi$$

$$x_3 = 8 - \theta - \varphi.$$

The solutions are all points with these two conditions. For example, one solution is $(s, t, u) = (1, 1, 1)$, then $x_1 = -2$ and $x_3 = 6$ giving us the solution $(x_1, x_2, x_3, x_4, x_5) = (-2, 1.6, 1, 1)$.

4 Calculus

To study functions we look at their behavior around local points. Three major concepts we will look at are limits, continuity, and differentiability. To formalize these concepts we use the (ϵ, δ) notation. We study the behavior of function $f(x)$ around a point $(x_0, f(x_0))$ and by bounding the differences $|\Delta f(x)| = |f(x) - f(x_0)| < \epsilon$ and $|\Delta x| = |x - x_0| < \delta$ we quantify it. If we can relate ϵ and δ such that for each ϵ (deviation in the function values) there exists a δ (interval on the x-axis) such that certain conditions hold.

4.1 Continuity

We have an intuitive notion that continuous functions are functions that do not jump in some sense. In this section, we are going to dive deeper into the idea of the continuity of real-valued functions.

To be more precise, we say that if we move the input value a little bit, then the output should move a little bit as well. Formally, we say that the function f is continuous at the point x_0 if, for every $\epsilon > 0$, there is $\delta > 0$ such that for all x:

$$|x - x_0| < \delta \to |f(x) - f(x_0)| < \epsilon$$

In words, we say that if the distance between x and x_0 is small, then the distance between the function evaluated at these points is also small. Moreover, if we fix a distance between function values (ϵ), we can find a sufficiently small distance (δ) such that all points (x) within the distance δ from x_0 that gives function values $f(x)$ that are within the distance ϵ from $f(x_0)$.

We show that the quadratic function $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = x^2$ is continuous at each point x_0 in its domain.

Intuitively, the function is continuous, see Figure 2 for a graphic interpretation.

So let x_0 be any real number. We need to prove that for each $\varepsilon > 0$ we can find a $\delta > 0$ such that points x with $|x - x_0| < \delta$ have a function value $f(x)$ with $|f(x) - f(x_0)| < \varepsilon$.

Since we need to show something for *each* $\varepsilon > 0$, it is customary to start with the sentence 'Let $\varepsilon > 0$': this means that we take a fixed, but arbitrary positive epsilon and use *only* that it is larger than zero: the argument will then apply to every positive epsilon, no matter what its specific value happens to be.

And we want to cleverly choose $\delta > 0$ — possibly depending on ε and x_0 — so that $|x - x_0| < \delta$ implies that $|f(x) - f(x_0)| < \varepsilon$. That is a lot easier if we find a relation between $|x - x_0|$ and $|f(x) - f(x_0)|$. Well,

$$\begin{aligned} |f(x) - f(x_0)| &= |x^2 - x_0^2| \\ &= |(x + x_0)(x - x_0)| \\ &= |x + x_0||x - x_0|. \end{aligned}$$

We know that the final term $|x - x_0|$ will be smaller than δ; the first term $|x + x_0|$ is still a bit tricky. Suppose for a while that we can find some upper bound $c > 0$ on this term, so that $|x + x_0| < c$. Together with $|x - x_0| < \delta$, we find

$$|f(x) - f(x_0)| = |x + x_0||x - x_0| < c\delta.$$

Remember: we want to make the expression above smaller than ε; we know it is smaller than $c\delta$. So if we choose δ to be at most ε/c, we are done.

One little problem remains: it presumes that we have an upper bound $c > 0$ on $|x + x_0|$ for our x with $|x - x_0| < \delta$. That's not so difficult. By the triangle inequality,

$$|x + x_0| = |(x - x_0) + 2x_0| \leq |x - x_0| + |2x_0|,$$

so if we make sure that, for instance, $|x - x_0|$ is less than 1, we have an upper bound $c = 1 + |2x_0|$:

$$|x + x_0| < 1 + |2x_0|.$$

Looking back through pur argument so far, we would like δ to be at most one (to have an upper bound $c = 1 + |2x_0|$ on the $|x + x_0|$ term) and at most ε/c to make the whole expression $|f(x) - f(x_0)|$ smaller than ε. So we could choose $\delta = \min\{1, \varepsilon/(1 + |2x_0|)\}$, and we are done!

After this rather lengthy argument, we can summarize the proof more concisely:

Proof. Let $\varepsilon > 0$. Choose

$$\delta = \min\left\{1, \frac{\varepsilon}{1 + |2x_0|}\right\}.$$

For each x with $|x - x_0| < \delta$, we have

$$|x + x_0| = |(x - x_0) + 2x_0| \leq |x - x_0| + |2x_0| < \delta + |2x_0| \leq 1 + |2x_0|,$$

so

$$|f(x) - f(x_0)| = |x + x_0||x - x_0| < (1 + |2x_0|)\delta < (1 + |2x_0|) \cdot \frac{\varepsilon}{1 + |2x_0|} = \varepsilon,$$

as desired. □

We take another example,

$$h(x) = \begin{cases} 5 - x & \text{if } -2 \leq x < 2 \\ 6 - x & \text{if } 2 \leq x \leq 5 \end{cases},$$

in Figure 3 we explore the $\epsilon - \delta$ properties for $h(x)$ in the point (2,2).

4.2 Limits

Next, we have limits of functions. Consider the function

$$f(x) = \begin{cases} 3 - x & \text{if } 1 < x < 2 \\ 2 & \text{if } x = 2 \\ x - 1 & \text{if } 2 < x \leq 3 \end{cases}$$

If we let x be close to 1, the function will be close to 2. We say that $lim_{x \to 1} f(x) = 2$. Similarly, $lim_{x \to 3} f(x) = 2$.

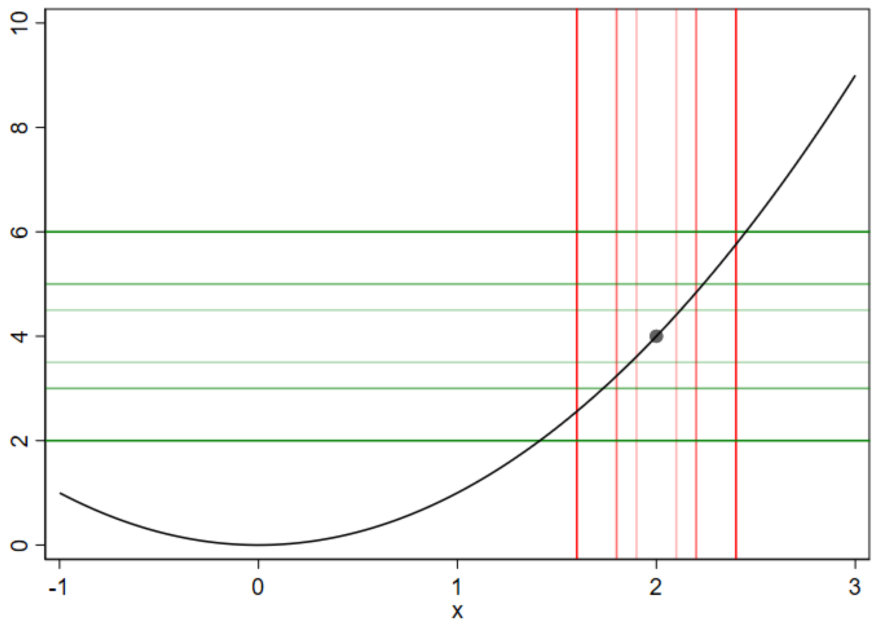

Figure 2: Function $g(x)$ is continius in the point (2,4). We see that $\epsilon - \delta$ holds for $\epsilon = \{2, 1, 0.5\}$ we choice $\delta = \{0.4, 0.2, 0.1\}$

We also note that f is not close to 2 in the area around 2, $f(2^+) \approx 1$ and $f(2^-) \approx 1$. We then say $lim_{x \to 2} f(x) = 1$, which is different from $f(2) = 2$ This is our intuition for f not being continuous at $x = 2$.

See Figure 4 for an illustration.

If all limits to x_0 of a function is the function value at x_0, ie $f(x_0)$ the function is continuous.

4.3 Differentiability

A related concept is differentiability. The derivative of a function measures how sensitive the output value is to the input value: if I consume one more apple, how much does my utility increase? In brief, we say that a function can be differentiated at a certain point whenever we can approximate it with a linear function. The formal definition is

$$\frac{df(x)}{dx} = lim_{h \to 0} \frac{f(x+h) - f(x)}{x + h - x}.$$

Now we look at some examples:

- Let's take our old example: $g(x) = x^2$. We know that it's derivative is

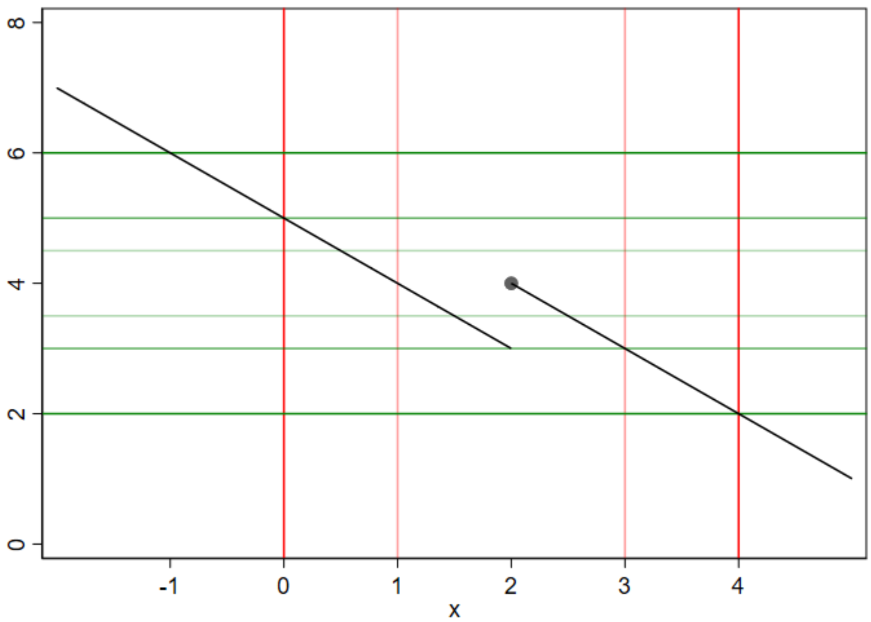

Figure 3: Function $g(x)$ is not continuous in the point (2,4). We see that $\epsilon - \delta$ holds for $\epsilon = \{2, 1\}$ were we choice $\delta = \{2, 1\}$. But for $\epsilon = 0.5$ no interval will capture the points to the left of $x = 2$. By contradiction, the function $h(x)$ is not continuous.

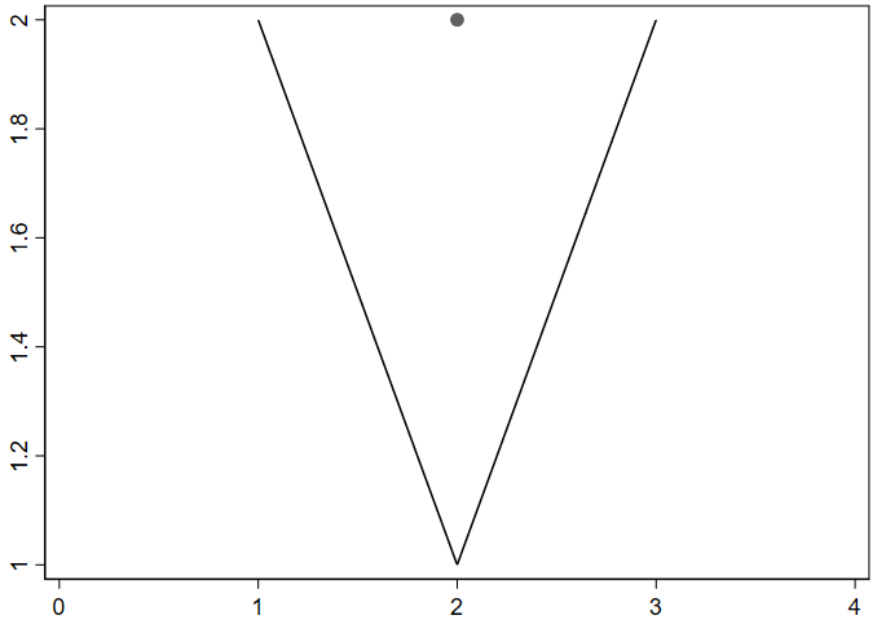

Figure 4: Function $f(x)$, as we see the limit does not exist, as the limit does not equal the value in the point. This corresponds to the function not being continuous.

$g'(x) = 2x$:

$$g'(x) = lim_{h\to 0}\frac{(x+h)^2 - x^2}{x+h-x} = lim_{h\to 0}\frac{x^2+2xh+h^2-x^2}{h} =$$

$$lim_{h\to 0}\frac{2xh+h^2}{h} = lim_{h\to 0}2x+h = 2x.$$

Since $2x$ is defined $\forall x : x \in \mathbb{R}$ we say it is continuous. But for example $ln(x)$ has the derivative $\frac{1}{x}$ which is not defined at $x = 0$

- Let's take a more difficult example, such as $sin(x)$. We know that its derivative is $cos(x)$. For this proof, we will use the three following statements without proof:
  - $sin(A+B) = sin(A)cos(B) + cos(A)sin(B)$
  - $lim_{h\to 0}\frac{sin(h)}{h} = 1$
  - $lim_{h\to 0}\frac{cos(h)-1}{h} = 0$

  Then:
$$\frac{d}{dx}sin(x) = lim_{h\to 0}\frac{sin(x+h)-sin(x)}{h} =$$
$$lim_{h\to 0}\frac{sin(x)cos(h)+cos(x)sin(h)-sin(x)}{h} =$$

$$lim_{h\to 0}\frac{sin(x)(cos(h)-1)+cos(x)sin(h)}{h} =$$

$$cos(x)[lim_{h\to 0}\frac{sin(h)}{h}] - sin(x)[lim_{h\to 0}\frac{1-cos(h)}{h}] =$$

$$cos(x) * (1) - sin(x) * (0) =$$

$$cos(x)$$

- We can also take the exponential function, given by e^t. Its derivative is e^t. For this proof, we will use the fact that $lim_{h\to 0}\frac{e^h-1}{h} = 1$

$$\frac{d}{dt}e^t = lim_{h\to 0}\frac{e^{t+h}-e^t}{h} =$$

$$lim_{h\to 0}\frac{e^t(e^h-1)}{h} =$$

$$e^t lim_{h\to 0}\frac{e^h-1}{h} =$$

$$e^t * 1$$

4.3.1 Gradients

So far, we've worked with derivatives of functions of only one variable. If our function has more variables, how should we think of derivatives? Consider the function $f(x, y) = x^2 + 2xy + y^2$ as our leading example.

Intuitively, we could imagine that we fix a value of y and take the derivative only with respect to x, treating y as if it were a constant. This is the intuition behind partial derivatives. The partial derivative of a function $f(x_1, ..., x_n)$ at a point $(a_1, ..., a_n)$ in the direction of variable x_i is:

$$\frac{\partial f}{\partial x_i}(a_1, ..., a_n) = lim_{h\to 0}\frac{f(a_1, ..., a_i + h, ..., a_n) - f(a_1, ..., a_n)}{h}$$

In our example, the partial derivative with respect to x is $2x + 2y$:

$$\frac{\partial f}{\partial x}(x_0, y_0) = lim_{h\to 0}\frac{f(x_0+h, y_0) - f(x_0, y_0)}{h} =$$

$$lim_{h\to 0}\frac{(x+h)^2 + 2(x+h)y + y^2 - x^2 - 2xy - y^2}{h}$$

$$lim_{h\to 0}\frac{x^2 + 2xh + h^2 + 2hy - x^2}{h} =$$

$$lim_{h\to 0}\frac{h(2x+h+2y)}{h} =$$

$$lim_{h\to 0}2x + h + 2y =$$

$$2x + 2y$$

Similarly, the partial derivative with respect to y is also $2x + 2y$.

Now, imagine a function that takes as arguments several variables and has a partial derivative for each one of them. Then, the partial derivatives at a point a defines a vector, $(\frac{\partial f}{\partial x_1}(a), ..., \frac{\partial f}{\partial x_n}(a))$. We call this vector the gradient and is often denoted as $\nabla f(a)$. In our example, $\nabla f(x_0, y_0) = (2x_0 + 2y_0, 2x_0 + 2y_0)$.

5 Proofs

This section is inspired by a note on proofs written by Thomas Seiler.

5.1 Propositions

While mathematics might seem like the arbitrary manipulation of curious symbols, it does obey certain internal rules. The study of these rules is the domain of (mathematical) logic. It is useful to understand the basics of logic to be able to understand and write proofs and internally coherent statements.

A basic concept in logic is propositions: these are statements that can be either true or false. "The car is red" is a proposition because it is true if the car is indeed red, and false otherwise. "The car is blah" is not a proposition because it does not make sense to ask if the car is "blah" unless we define blah to mean something.

We can also negate propositions: The negation of "It rains" is "It does not rain"; if we call the former proposition p, then its negation is $\neg p$. The negation is true whenever the original proposition is false, and vice versa.

5.2 Binary Operations

If we have (at least) two propositions, we can start combining them. There are some basic ways of doing this, which we will go through below. For the example, consider two arbitrary propositions p and q:

- Negation ($\neg p$): The negation is true if and only if the statement p is false. In ordinary speech, it refers to "NOT", "I am not an economist" is true if I am something else than an economist[1].

- Conjunction ($p \wedge q$): The conjunction is true if and only if both p and q are true at the same time, otherwise it is false. In ordinary speech, it refers to "AND". "I am cold and wet" is true if I am indeed cold and wet.

- Disjunction ($p \vee q$): The disjunction is true if p or q is true, or both. In ordinary speech, this refers to "OR". "I want to travel to Norway or Finland" is true if I want to travel to at least one of them.

- Implication ($p \rightarrow q$): The implication is true says that q is true if p is true. If p is false, we say that the implication is true independent of the truth value of q. In ordinary speech, it somewhat resembles "if-then" statements. "If it rains I get wet" is false only if it rains and I don't get wet.

- Equivalence ($p \iff q$): The equivalence is true if p and q has the same truth values. In ordinary speech, it somewhat resembles statements of the form "if and only if". "I get to Lyon if and only if I catch the train".

[1] You can express all five binary operators as a combination of the negation and one of the four others. If you are bored show this and email me.

Table 1: Truth Table

p	q	$p \wedge q$	$p \vee q$	$p \rightarrow q$	$p \iff q$
T	T	T	T	T	T
T	F	F	T	F	F
F	T	F	T	T	F
F	F	F	F	T	T
Syntax		AND	OR	if-then	iff

It can be tricky to keep track of when the different combined statements are true. A helpful device is a truth table. Truth tables start by listing whether the propositions are true or false and then show if the different binary operations are true or false. In table 1 we see this for the four operations above.

We can use these rules to do a logical proof. Start by listing the assumptions regarding propositions you want, then use the relations between propositions to deduce new propositions:

- Premise 1: p
- Premise 2: $p \rightarrow q$
- Using premises 1 and 2 and "implication": q
- Conclusion: $p \wedge q$

5.3 Common Proofs

5.3.1 Direct Proof

The direct proof works by directly verifying the implication in question, i.e. start with premises $p_1, ..., p_n$ and then show that q does indeed follow. Example: "Show there is no integer in the set $A = \{\pi, e, \sqrt{2}\}$". Then we loop through the set: π is not an integer, e is not an integer and $\sqrt{2}$ is not an integer. Thus, there are no integers in the set.

5.3.2 Proof by Contradiction

Proofs by contradiction rely on the fact that a proposition cannot be true and false at the same time. If we want to show that p is true, it is the same as showing $\neg p$ is false. We assume p is false and then show that we reach a contradiction somewhere and then deduce that p must be true.

One of the most comon examples is the proof that $\sqrt{2} \in \mathbb{R} \quad \& \quad \sqrt{2} \notin \mathbb{Q}$. It dates back to Aristotle and Euclides. Here is a short sketch, that takes inspiration from *Principal of Mathematical Analysis* by Walter Rudin.

Theorem 1. $\sqrt{2} \in \mathbb{R} \quad \& \quad \sqrt{2} \notin \mathbb{Q}$

Proof. Using the method proof by contradiction we assume that the statement is false, ie

$$\sqrt{2} \in \mathbb{Q}.$$

Any rational number p can be written as $p = \frac{m}{n}$ in a unique way, where $\frac{m}{n}$ are irreducible (they have no prime factor in common). Assume that the two integers mn represent this unique irreducible representation. Then

$$\sqrt{2} = \frac{m}{n} \implies 2 = \frac{m^2}{n^2} \implies 2n^2 = m^2.$$

As any number multiplied by 2 is even, m^2 is even. Then m is also even (if m were odd, m^2 would be odd), and so m^2 is divisible by 4. Let $m^2 = 4k$ for some integer k, then $n^2 = 4k$, so n^2 is even, and so is n as well. Since any even number has 2 as a prime factor, m and n share a prime factor, contradicting the assumption that $\frac{m}{n}$ is irreducible. Therefore p is not a rational number.

□

5.3.3 Proof by Induction

Proof by induction works as follows. Assume we have a sequence of statements, $p(n)$, with $n \in \mathbb{N}$. Then, if we show that the statement is true at $n+1$ if it is true at n and that it is true at the start $p(1)$, we have shown it is true for all n.

As an example, we will show that the sum of the first n odd numbers is given by n^2. We begin by showing that it's true for $n = 1$ and then show that it holds at $n+1$ of it holds at the nth step.

If $n = 1$, we simply have $1 = 1^2$. So it holds in this case.

Assume $\sum_{k=1}^{n}(2k-1) = n^2$, then:

$$\sum_{k=1}^{n+1}(2k-1) = n^2 + 2(n+1) - 1 = n^2 + 2n + 1 = (n+1)^2$$

Which is what we wanted to show.

5.3.4 Contrapositive Proof

Another proof is the contrapositive proof, which exploits the fact that to show $p \to q$, we can as well show $\neg q \to \neg p$. It might not be obvious that they are equivalent, but we can use a truth table to see that they are true and false at the same time (table 2).

Table 2: Another Truth Table

p	q	$p \to q$	$\neg q \to \neg p$
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

We can use this method to show the statement if x^2 is even, then x is even. The contrapositive statement is: if x is not even, then x^2 is not even. Then we use a direct proof to show this.

Assume x is not even, then $x = 2k + 1$ for some k.

Then, $x^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 4k(k + 1) + 1$.

Notably, $4k(k + 1)$ is even, which means $4k(k + 1) + 1$ is not even.

This finishes the proof.

5.4 On Definitions

Later in the course, we will define various concepts. We will write these on the form "X is Y if certain conditions hold". We will not write it on the form "X is Y if and only if certain conditions hold", because Y is not defined before we introduce the definition, so we cannot assign any truth value to it. However, once it is defined, we can use it as any other proposition.

6 Concluding Words: The Game is Set

On these few pages, we've seen some of the building blocks of mathematics. While it might be difficult to piece everything together, hopefully, you will soon realize that it's all about taking small pieces and putting them together in more and more complex ways. Just like chess games can be very complicated, it all stems from the rules of how to move the different pieces.

I hope you enjoyed this "boot camp" and that you feel more prepared for the Master's program in Economics. I hope you have two great years at SSE!